arXiv:hep-th/9605082v2 16 Sep 1996

Vertical Versus Diagonal Dimensional Reduction for p-branes

H. Lü [†], C.N. Pope [†]

Center for Theoretical Physics, Texas A&M University, College Station, Texas 77843

K.S. Stelle [⋆]

The Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, UK

ABSTRACT

In addition to the double-dimensional reduction procedure that employs world-volume Killing symmetries of p-brane supergravity solutions and acts diagonally on a plot of p versus spacetime dimension D, there exists a second procedure of "vertical" reduction. This reduces the transverse-space dimension via an integral that superposes solutions to the underlying Laplace equation. We show that vertical reduction is also closely related to the recently-introduced notion of intersecting p-branes. We illustrate this with examples, and also construct a new $D = 11$ solution describing four intersecting membranes, which preserves 1/16 of the supersymmetry. Given the two reduction schemes plus duality transformations at special points of the scalar modulus space, one may relate most of the p-brane solutions of relevance to superstring theory. We argue that the maximum classifying duality symmetry for this purpose is the Weyl group of the corresponding Cremmer-Julia supergravity symmetry $E_{r(+r)}$. We also discuss a separate class of duality-invariant p-branes with $p = D - 3$.

1 Introduction

In this paper, we shall explore the two different types of dimensional reduction that may be used to relate p-brane solutions of supergravity theories in different dimensions. The better-known of these types of reduction employs Killing symmetries of the p-bane solutions in D dimensions to effect a simultaneous reduction on the world-volume and in the target space, yielding a $(p-1)$-brane in $(D-1)$ dimensions. This is thus the field-theoretic analogue of the double dimensional reduction procedure for p-brane *actions* [1], which is also applicable to supergravity *solutions*, causing "diagonal" movement on the D versus p "brane-scan." Aside from a Weyl rescaling of the metric needed to maintain the conventional Einstein-frame form of the gravitational action, this diagonal reduction procedure does not change the asymptotic falloff behaviour of a solution in the directions transverse to the world-volume.

The second type of reduction has been referred to in the literature as "constructing periodic arrays" [2, 3, 4]. This procedure employs the zero-force property of p-brane solutions, which permits multi-center solutions as well as single-center ones. We shall present, in some more detail than previously given, just what this second procedure entails. In particular, it requires the possibility of stacking up a "deck" of an infinite number of p-branes in D dimensions, so that the sum of potentials in a multi-center solution goes over to an integral, thus changing the asymptotic falloff behaviour in the transverse directions to that corresponding to a p-brane in $D-1$ dimensions. Thus, we shall call this "vertical" dimensional reduction, corresponding to vertical movement on the brane-scan. In contrast to some impressions gathered from the literature, we shall show that while the zero-force properties of supersymmetric p-branes indeed allow this type of reduction, there are in fact many more non-supersymmetric but extremal solutions that also satisfy the no-force condition. Vertical dimensional reduction can also be applied to these cases.

This stacking-up procedure breaks down for the case of $(D-3)$-branes in D dimensions, which have conical asymptotic spacetimes in which each increment to the number of centers in a multi-center solution causes a portion of the solid angle at transverse infinity to be removed, eventually leading to a compactification of the transverse spacetime. Thus, the known examples [5, 6, 7] of $(D-2)$-branes seem to reside in a different category from the other p-branes from this perspective. This is due in part to the fact that they occur in different theories [8]: instead of the standard supergravity theories that we consider here, they require a dilaton potential term, which may be interpreted as the prefactor for a zero-form field strength [7].

The vertical reduction procedure involves two stages, namely an integration over a

continuum of charges of the multi-center solution distributed uniformly over lines, planes or hyperplanes, followed by an ordinary Kaluza-Klein reduction over the resulting Killing directions. We show that at the intermediate stage, the metric configuration in the higher dimension can sometimes be interpreted as a special case of intersecting p-branes. Thus we see that some p-brane solitons that are vertical-dimensional reductions of p-branes in higher dimensions can also be viewed as intersections of extended objects in the higher dimension. In particular, all the multi-scalar and dyonic solutions whose charges are carried by field strengths derived from the field strengths of the higher-dimensional theory admit such an interpretation. In practice, the higher dimension of greatest interest is $D = 11$, in which case these lower-dimensional p-branes can be interpreted as intersections of M-branes.

The two types of dimensional reduction can be combined with duality transformations in dimensions where the vertical and diagonal reduction trajectories cross. For this purpose, it is necessary to focus on the subgroup of duality transformations that map between *distinct* p-brane solutions, *i.e.* the classifying symmetry for p-brane solutions. We shall argue that the maximal relevant duality group here is the Weyl group of the Cremmer-Julia symmetry $E_{r(+r)}$, where $r = (11 - D)$, as discussed in [9]. In the case of vanishing asymptotic values of the scalar fields, this is the subgroup of the U-duality group [10] $E_{r(+r)}(\mathbb{Z})$ that leaves unchanged the scalar moduli of the theory, *i.e.* the asymptotic values of the scalar fields. In other words, the Weyl group is the U-duality little group of the scalar vacuum. In analogy to the case of asymptotic Poincaré symmetry in general relativity with a flat asymptotic metric, it seems natural to interpret this as a rigid, solution-classifying symmetry, instead of as a "local" symmetry describing different forms of identified solutions. If duality is combined with the two forms of dimensional reduction at special points in the scalar modulus space in this way, it welds together into one large family many of the p-branes discussed in connection with string theory.[1] For example, by combining vertical and diagonal dimensional reduction with a simple Weyl-group duality transformation, one may link the string and 5-brane solutions of $D = 10$ supergravity.

The interplay between the two types of dimensional reduction and duality multiplets suggests the question whether there are duality singlet representations. The standard p-brane solutions form non-trivial duality multiplets. We shall show, however, that at least in the case of $(D-3)$-branes, there is a class of solutions that are invariant under an $SL(2, \mathbb{R})$

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[1]This family excludes so far the $(D-2)$ branes, as discussed above. It may be relevant in this connection that $(D-3)$ branes are dual to supersymmetric instantons, while $(D-2)$-branes do not seem to have sensible duals.

duality symmetry. Whether such a construction can be extended to solutions invariant under full duality groups remains an open problem.

A p-brane solution of a D-dimensional supergravity theory is a configuration describing a flat $d = (p + 1)$ dimensional Minkowski world volume. The D-dimensional metric takes the form

$$ds^2 = e^{2A}dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B}dy^m dy^m, \tag{1.1}$$

where A and B are functions only of the y^m coordinates of the $(D-d)$-dimensional transverse space. The relevant part of the Lagrangian, containing the metric, a dilatonic scalar and an n-index field strength, is

$$\mathcal{L} = eR - \tfrac{1}{2}(\partial\phi)^2 - \frac{1}{2n!}e^{a\phi}F_n^2 . \tag{1.2}$$

Here the constant a can be parameterised by

$$a^2 = \Delta - \frac{2d\tilde{d}}{D-2} . \tag{1.3}$$

where $\tilde{d} \equiv D - d - 2$. This Lagrangian can give rise to either elementary p-brane solutions with world volume dimension $d = n - 1$, or to solitonic solutions with $d = D - n - 1$. In the former case, F_n carries an electric-type charge; in the latter case, it carries a magnetic type charge. Of particular interest are isotropic solutions where the functions A and B and the dilatonic scalar field ϕ depend only on the variable $r = \sqrt{y^m y^m}$. These solutions take the form [11]

$$\begin{aligned}
ds^2 &= \left(1 + \frac{k}{r^{\tilde{d}}}\right)^{-\frac{4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^{\tilde{d}}}\right)^{\frac{4d}{\Delta(D-2)}} dy^m dy^m , \\
e^\phi &= \left(1 + \frac{k}{r^{\tilde{d}}}\right)^{\frac{2a}{\epsilon\Delta}} ,
\end{aligned} \tag{1.4}$$

where $\epsilon = 1$ for elementary solutions and $\epsilon = -1$ for solitonic solutions. The solution (1.4) is extremal, and has a mass per unit p-volume $m = \frac{\lambda}{2\sqrt{\Delta}}$, and from the field strength F_n one finds a charge $\frac{\lambda}{4}$, where $\lambda = -2\tilde{d}k/\sqrt{\Delta}$. It describes a p-brane with a single center located at $r = 0$. The value of Δ for the 4-form field strength in $D = 11$ is 4, corresponding to $a = 0$, which reflects the fact that there is no dilaton in the theory. On descending through the dimensions, the possible values of Δ proliferate. When $\Delta = 4/k$, supersymmetric solutions can arise [12]. However, the vast majority of allowed values correspond to extremal solutions that are not supersymmetric [12].

Note that in this solution, the functions A, B and the dilaton ϕ are linearly related, namely

$$dA + \tilde{d}B = 0 , \qquad \phi = -\frac{\epsilon(D-2)a}{\tilde{d}}A . \tag{1.5}$$

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It was shown in [11] that a single-scalar isotropic p-brane in D dimensions can be double-dimensionally reduced to a single-scalar $(p-1)$-brane in $(D-1)$ dimensions, with the same value of Δ. This is perhaps surprising, since the Kaluza-Klein procedure introduces an additional scalar field. The fact that a linear combination of this and the original scalar of the D-dimensional solution vanishes is a non-trivial consequence of the precise coefficients in the relations (1.5). Indeed, there are other kinds of p-brane solution where these relations are not satisfied. For example, in the dyonic string in $D = 6$, the function A and the dilaton ϕ are independent. As a consequence, the Kaluza-Klein procedure gives rise to a two-scalar black hole [13] in $D = 5$, which carries two independent charges. If either the electric or the magnetic charge of the dyon vanishes, the corresponding two-scalar black hole in $D = 5$ reduces to a single-scalar one with $\Delta = 4$. On the other hand, if both charges are equal, it also reduces to a single-scalar black hole, but with $\Delta = 2$. It was shown in [9] that in the context of maximal supergravity with vanishing scalar moduli, the two-scalar black holes in $D = 5$ form a 270-dimensional representation of the Weyl group of the U duality group $E_6(\mathbb{Z})$. We have seen that some members of this multiplet oxidise to isotropic dyonic strings in $D = 6$. Others, on the other hand, oxidise to boosted strings in $D = 6$, which carry either electric or magnetic charge on a 3-form field strength, together with a momentum associated with the charge carried by the Kaluza-Klein 2-form. Another example where the ansätze (1.5) are relaxed is provided by the isotropic single-scalar non-extremal black p-branes discussed in [14]; these are the general solutions of the equations of motion. These involve non-trivial solutions of the Liouville or Toda equations. Double-dimensional reduction does not establish relations between solutions of this type of relaxed ansatz in different dimensions.

2 Multi-center solutions

Now, we shall construct multi-center solutions of the supergravity theories. We begin with the elementary case. The metric ansatz for a multi-center solution is once more given by (1.1), and again A and B will be taken to be functions of the transverse coordinates y^m. The ansatz for the n-form field strength is

$$F_{m\mu_1\cdots\mu_{n-1}} = \epsilon_{\mu_1\cdots\mu_{n-1}}\partial_m e^C \ , \tag{2.1}$$

where C is taken to be a function of the transverse space coordinates y^m. It is straightforward to obtain the equations of motion following from the Lagrangian (1.2) and to substitute the ansätze for the field strength and the metric. As in the case of single-center solutions,

one may obtain simple solutions by making a further ansatz, namely $dA + \tilde{d}B = 0$. Here, however, we do not assume isotropicity in the transverse space. In order to obtain multi-center solutions, we shall first show that the resulting equations of motion can be cast into a linear form. The remaining equations of motion are given by

$$\partial_m \partial_m \phi = -\tfrac{1}{2}aS^2 \ , \qquad \partial_m \partial_m A = \tfrac{\tilde{d}}{2(D-2)}S^2 \ ,$$
$$d(D-2)(\partial_m A)^2 + \tfrac{1}{2}\tilde{d}(\partial_m \phi)^2 \ = \ \tfrac{1}{2}\tilde{d}S^2 \ , \tag{2.2}$$

where

$$S^2 = e^{a\phi - 2dA}(\partial_m e^C)^2 \ , \tag{2.3}$$

together with an equation of motion for the field strength F_n, which we shall discuss later. The equations (2.2) can be solved by taking $\phi = -a(D-2)A/\tilde{d}$, which implies that

$$S^2 = \Delta(\partial_m \phi)^2/a^2 \ , \tag{2.4}$$

and

$$\partial_m \partial_m \phi + \tfrac{\Delta}{2a}(\partial_m \phi)^2 = 0 \ . \tag{2.5}$$

This can be rewritten as the linear Laplace equation

$$\partial_m \partial_m e^{\frac{\Delta}{2a}\phi} = 0 \ , \tag{2.6}$$

whose point-charge solutions can be superposed to yield multi-center solutions. The function C may be found by combining (2.3) and (2.4), which may be solved by setting $\partial_m e^C = (\sqrt{\Delta}/a)\,\partial_m \phi\, e^{-\frac{1}{2}a\phi + dA}$. It is easy to verify that this is consistent with the equation of motion for the field strength F_n:

$$\partial_m \partial_m C + \partial_m C(\partial_m C + \tilde{d}\partial_m B - d\partial_m A + a\partial_m \phi) = 0 \ . \tag{2.7}$$

Thus the elementary multi-center p-brane solution is given by

$$ds^2 \ = \ \Big(1 + \sum_\alpha \frac{k_\alpha}{|\vec{y} - \vec{y}_\alpha|^{\tilde{d}}}\Big)^{-\frac{4\tilde{d}}{(D-2)\Delta}} dx^\mu dx^\nu \eta_{\mu\nu} + \Big(1 + \sum_\alpha \frac{k_\alpha}{|\vec{y} - \vec{y}_\alpha|^{\tilde{d}}}\Big)^{\frac{4d}{(D-2)\Delta}} dy^m dy^m \ ,$$
$$e^{\frac{\Delta}{2a}\phi} \ = \ 1 + \sum_\alpha \frac{k_\alpha}{|\vec{y} - \vec{y}_\alpha|^{\tilde{d}}} \ , \qquad e^C = \tfrac{2}{\sqrt{\Delta}} e^{-\frac{\Delta}{2a}\phi} \ . \tag{2.8}$$

The above is the generic elementary solution for $\tilde{d} \neq 0$. When $\tilde{d} = 0$, the solution (2.8) is modified by the replacement of $|\vec{y} - \vec{y}_\alpha|^{-\tilde{d}}$ by $\log|\vec{y} - \vec{y}_\alpha|$. In fact when $\tilde{d} = 0$, one can also build a modular-invariant p-brane solution, which we shall discuss in section 4. The electric charge and the mass per unit p-volume for the solution (2.8) are given by $Q = \sum_\alpha \lambda_\alpha/4$ and $m = \sum_\alpha \lambda_\alpha/(2\sqrt{\Delta})$, where $\lambda_\alpha = -2\tilde{d}k_\alpha/\sqrt{\Delta}$.

Given the above explicit form for elementary solutions, it is now straightforward to construct solitonic solutions. The solitonic ansatz for the field strength is given by

$$F_{m_1 \cdots m_n} = -\epsilon_{m_1 \cdots m_n p} \, \partial_p \sum_\alpha \frac{\lambda_\alpha / \tilde{d}}{|\vec{y} - \vec{y}_\alpha|^{\tilde{d}}} \; . \tag{2.9}$$

The solitonic solution can then be straightforwardly constructed, and is in fact given by (2.8) after the replacement $\phi \longrightarrow -\phi$.

Note that we have obtained here generic solutions for all values of Δ. The values of Δ for supersymmetric p-brane solutions are given by $\Delta = 4/k$, where k is the number of participating field strengths. Thus one can construct multi-center solutions even for non-supersymmetric cases, providing a counter-example to the commonly-held belief that the no-force condition is satisfied only by supersymmetric solutions.

3   Vertical dimensional reduction

As we have seen above, dimensional reduction of a p-brane solution to a supergravity theory can be implemented in two quite different ways. The usual process of Kaluza-Klein dimensional reduction has the effect of lowering the p-brane world volume dimension at the same time. Thus in this double dimensional reduction procedure, a p-brane solution in D dimensions can be reduced on the x^μ coordinates to a $(p-1)$-brane solution in $(D-1)$ dimensions. This keeps the dual dimension $\tilde{d} = D - d - 2$ at a constant value. By contrast, vertical dimensional reduction works on the transverse coordinates y^m, reducing D while preserving the dimension of the p-brane world volume, thus reducing $\tilde{d}$ by one but keeping d constant.

The two dimensional-reduction procedures are implemented in very different ways. This difference stems from the fact that the fields in the p-brane solutions are functions of the y^m coordinates of the transverse space, but are independent of the world-volume coordinates x^μ. Thus, the diagonal process of Kaluza-Klein double-dimensional reduction, in which the fields are taken to be independent of one of the x^μ coordinates, has no essential effect on the structure of the solutions. Essentially, one is reducing on a coordinate on which the solution does not depend anyway. (The metric does in fact undergo a conformal rescaling, but this is only because in Kaluza-Klein dimensional reduction, one customarily rescales the metric so as to remain in the Einstein frame in the lower dimension.) Thus under double dimensional reduction, all fields in $(D-1)$ dimensions retain the same form as they had in D dimensions, with d replaced by $(d-1)$. In particular, $\tilde{d}$ and Δ, and hence the asymptotic behaviour at infinity, remain unchanged.

In the vertical dimensional-reduction procedure, it is $\tilde{d}$ rather than d that is reduced. Consequently, as can be seen from (1.4), the asymptotic behaviour of the fields must change. This can be achieved because the equations governing p-brane solutions can be cast into a linear form, and consequently they admit multi-centered solutions in which single-center solutions are superposed. If we align the centers of a uniform continuum of single-centered solutions along some axis in the transverse space, then the resulting solution acquires a translational symmetry in that direction and can thus be interpreted as a solution with one transverse-space dimension fewer. This is analogous to the construction of an electrostatic potential in two spatial dimensions by superposing the solutions for a continuous line of point charges in three spatial dimensions. An important difference, however, is that the multi-centered p-brane solutions have a zero-force condition, which is necessary for static equilibrium. It is sometimes argued that supersymmetry of the solution is responsible for this zero-force condition. However, as we have seen in the previous section, static multi-centered solutions are also possible for non-supersymmetric configurations.

In giving the details of the vertical dimensional reduction procedure, it is convenient to consider first the case with $\tilde{d} \geq 2$. We take a continuous stack of p-branes with uniform charge density along the coordinate $z = y^{D-d}$. It follows that the sum over discrete charge centers is replaced by an integral,

$$\sum_{\alpha} \frac{k_{\alpha}}{|\vec{y} - \vec{y}_{\alpha}|^{\tilde{d}}} \longrightarrow \int_{-\infty}^{\infty} \frac{kdz}{(\tilde{r}^2 + z^2)^{\frac{\tilde{d}}{2}}} = \frac{\tilde{k}}{\tilde{r}^{\tilde{d}-1}} \,, \tag{3.1}$$

where $\tilde{k} = k\sqrt{\pi}\Gamma(\tilde{d} - \frac{1}{2})/(2\Gamma(\tilde{d}))$, and $\tilde{r}^2 = y^{\tilde{m}}y^{\tilde{m}} = (y^1)^2 + \cdots (y^{D-d-1})^2$. The metric of the p-brane solution then becomes

$$ds^2 = \left(1 + \frac{\tilde{k}}{\tilde{r}^{\tilde{d}-1}}\right)^{-\frac{4\tilde{d}}{\Delta(D-2)}} dx^{\mu}dx^{\nu}\eta_{\mu\nu} + \left(1 + \frac{\tilde{k}}{\tilde{r}^{\tilde{d}-1}}\right)^{\frac{4d}{\Delta(D-2)}} (dz^2 + dy^{\tilde{m}}dy^{\tilde{m}}) \,. \tag{3.2}$$

Since this solution is now independent of the coordinate $z = y^{D-d}$, one may project it into the $(D-1)$-dimensional subspace. The resulting metric is not yet the usual one (1.4) for a p-brane in $(D-1)$ dimensions. The difference, however, is merely accounted for by scaling with an overall conformal factor. In other words, the $(D-1)$-dimensional solution is obtained in a frame that is not the usual Einstein frame. If we multiply the above metric by a conformal factor $(1 + \tilde{k}\tilde{r}^{-\tilde{d}+1})^{4d/(\Delta(D-2)(D-3))}$, the solution takes precisely the form (1.4) in $(D-1)$ dimensions, with the *same* value of Δ as for the original solution in D dimensions. Thus if one aligns a continuum of p-branes with uniform charge density along an axis in the transverse space, the spacetime configuration is effectively reduced to $(D-1)$ dimensions, since it is now uniform in the direction of this axis. From the

point of view of the remaining coordinates, the solution is effectively a p-brane in $(D-1)$ dimensions. As in the case of double dimensional reduction, the value of Δ is preserved by the vertical dimensional reduction procedure. This is consistent with the fact that vertical dimensional reduction preserves the supersymmetry of the solution. This preservation of supersymmetry occurs because the conditions for unbroken supersymmetry depend on the algebraic relations between the functions A, B, C and ϕ, and not on the particular single-center (1.4) or multi-center (2.8) form of the solution to the Laplace equation (2.6)

The process of vertical dimensional reduction described above can be thought of as a two-stage procedure. First, by integrating over a continuum of single-center solutions along a line in the transverse space, the metric is reduced to the form (3.2). As yet, no reduction in the total spacetime dimension has taken place. Although the metric is now independent of z, this coordinate cannot simply be included along with the p-brane world-volume coordinates x^μ to describe a $(p+1)$-brane, since the functional dependence of its $\tilde{r}$-dependent prefactor differs from that for the world-volume coordinates. Nevertheless, the coordinate z is a world-volume-like coordinate, since the solution is independent of z. In this first stage, the integration over z gives rise to a new Killing vector, which generates translations along z. We can then, as a second step, perform the standard Kaluza-Klein reduction on the coordinate z, thereby obtaining a p-brane in $(D-1)$ dimensions. In this example, we considered the case where the charge is uniformly distributed along a line parameterised by the coordinate z. To perform vertical dimensional reduction to lower dimensions, we can apply this procedure iteratively, or we can directly consider the cases where the charges are uniformly distributed on a plane or a hyperplane, on which the standard Kaluza-Klein procedure can then be performed.

In cases with the initial value $\tilde{d} = 1$, the integration (3.1) becomes divergent. These cases can be handled by putting a cutoff and renormalising by subtracting an infinite constant; the result for the analogue of (3.1) is of the form $(1 + k \log r)$, which is a solution to the Laplace equation in two dimensions, corresponding to $\tilde{d} = 0$. Thus the $(D-4)$-brane in D dimensions reduces to a $(D-4)$-brane in $(D-1)$ dimensions, *i.e.* it becomes a member of the $(D-3)$-brane diagonal trajectory. This appears to be the last sensible step for vertical reduction, however, as the $(D-2)$-branes that lie on the next lower diagonal trajectory exist [5, 6, 7] only in a different class of supergravity theories [8] that include a dilatonic-scalar potential. These dilatonic potentials generalise the cosmological term in the action, and have the effect of ruling out empty flat space as a solution.[2]

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[2]It is true that such supergravity theories have recently been given a common formulation together with

The difficulties in proceeding with vertical dimensional reduction from $(D-3)$-branes to $(D-2)$-branes is mirrored by a special aspect of the asymptotic spacetimes for $(D-3)$-branes. The transverse asymptotic spaces of $(D-3)$-branes are not asymptotically flat, but are instead asymptotically conical, with a deficit angle related to the ADM energy density of the solution. Although $(D-3)$-branes do obey the zero-force property, any attempt to stack up enough of them along an axis so as to generate the transverse translational symmetry required for vertical reduction runs into the difficulty that the total D-dimensional mass density then exceeds the limit at which the transverse spacetime is "eaten up" by the deficit angle bites taken out of it. The $(D-3)$-branes thus appear to be the natural endpoints of the process of vertical dimensional reduction.

4   Vertical reduction and intersecting p-branes

In double-dimensional reduction, an isotropic p-brane solution in $(D-1)$ dimensions can often be viewed as an isotropic $(p+1)$-brane in D dimensions. On the other hand, in vertical-dimensional reduction, an isotropic p-brane in $(D-1)$ dimensions can no longer be isotropically oxidised into D dimensions (hence the terminology "stainless" introduced in [11]), but as we saw in the previous section it can be viewed as a p-brane solution in D dimensions whose charge is uniformly distributed along the extra coordinate. Configurations of this type are closely related to the recently-introduced notion of intersecting p-branes [15, 16, 17, 18]. To illustrate this, let us consider the two-scalar black hole [13] in $D = 7$, with two independent electric charges q_{12} and q_{34} carried by the 2-form field strengths $F^{(12)}$ and $F^{(34)}$. These field strengths $F^{(ij)}$ arise from dimensional reduction of the 4-form field strength F_{MNPQ} in $D = 11$, in which two of the indices lie in the internal directions z_i and z_j. This black hole can be dimensionally oxidised to $D = 11$, giving rise to two intersecting membranes in $D = 11$ supergravity, or M-theory. The two membranes share a common time coordinate, but have two orthogonal spatial surfaces, namely (z_1, z_2) and (z_3, z_4) respectively. The 11-dimensional metric is given by [16, 18]

$$
\begin{aligned}
ds^2 = & -H_{12}^{-2/3} H_{34}^{-2/3} dt^2 + H_{12}^{1/3} H_{34}^{-2/3} (dz_1^2 + dz_2^2) \\
& + H_{34}^{1/3} H_{12}^{-2/3} (dz_3^2 + dz_4^2) + H_{12}^{1/3} H_{34}^{1/3} (dy_1^2 + \cdots + dy_6^2) \,,
\end{aligned}
\tag{4.1}
$$

the standard ones by the introduction of a rank $D-1$ antisymmetric-tensor gauge field [6], which has no continuous degrees of freedom but which introduces the coefficient of the dilaton-scalar potential as an integration constant. It is not clear to us if this will be helpful in pushing vertical dimensional reduction beyond the $(D-3)$-brane barrier.

where $H_{ij} = 1 + q_{ij}/r^4$, and $r^2 = y_1^2 + \cdots y_6^2$. This metric can be interpreted as describing two intersecting membranes [16, 18]. The reason for this is as follows. If we consider the limit where $q_{12} = 0$, the metric then describes a multi-center membrane [19], with world-volume coordinates (t, z_1, z_2), in which the multiple charges are distributed uniformly over the (z_3, z_4) plane. On the other hand, if q_{34} is instead set to zero, the metric describes a different multi-center membrane with world-volume coordinates (t, z_3, z_4), with its charges distributed uniformly over the (z_1, z_2) plane. If we now consider the general case where q_{12} and q_{34} take generic non-vanishing values, we see that the solution (4.1) interpolates between these two limiting cases, and can be interpreted as describing the intersection of the two membranes. Note that *a priori*, neither of the limiting cases $q_{12} = 0$ or $q_{34} = 0$ by itself would deserve the interpretation as an intersection of membranes, because the (z_3, z_4) or (z_1, z_2) planes respectively only acquire an interpretation as spatial membrane world-volume coordinates when one moves away from the limiting cases. It is the ability to interpolate between the two limits that provides the compelling interpretation as an intersection of membranes.

Another limiting case that is of interest is when the two charges q_{12} and q_{34} are equal. In this case, the the resulting $\Delta = 2$ black hole in $D = 7$ becomes a Güven solution [20] in $D = 11$, which was interpreted as a special case of two intersecting membranes in [15]. Again, if one had only this special example of the Güven solution, where $H_{12} = H_{34}$, the interpretation as intersecting membranes would be rather obscure, since in this degenerate limit, there is nothing that breaks the symmetry of the four coordinates z_1, z_2, z_3 and z_4. Thus again it seems to be important for this interpretation of the Güven solution that there should exist more general solutions where the harmonic functions H_{12} and H_{34} can be independent.

Having established the interpretation of the class of solutions given by (4.1), it is of interest to note that the two limiting cases $q_{12} = 0$ or $q_{34} = 0$ can be viewed as the first stage of a vertical dimensional reduction of the membrane in $D = 11$ to a membrane in $D = 9$. Put another way, we see that the $q_{12} = 0$ or $q_{34} = 0$ limits of the solutions (4.1) are the Kaluza-Klein oxidations of certain membrane solutions in $D = 9$. Thus vertical dimensional reduction provides a way of interpreting certain lower-dimensional p-branes as intersections of extended objects in eleven-dimensional M-theory.

It should be emphasised, however that not all stainless p-brane solitons can be viewed as intersecting branes in $D = 11$. First of all, the charges of the p-brane in the lower dimension must all be carried by field strengths that are derived from the 4-form in $D = 11$ rather than

from the Kaluza-Klein reduction of the metric, since otherwise the oxidation will generate off-diagonal "boosts" in the the $D = 11$ metric. (In fact, these types of p-branes can be viewed as boosted intersecting M-branes [21].) Secondly, even if the solution does satisfy this condition, it is not always possible to interpret it as intersecting p-branes in $D = 11$. For example, the membrane in $D = 10$ can be viewed as the multi-center membrane in $D = 11$ with the charges lying along the extra dimension z,

$$ds^2 = H^{-2/3}(-dt^2 + dx^i dx^i) + H^{1/3}(dy^m dy^m + dz^2) . \tag{4.2}$$

At first sight, one might interpret this in $D = 11$ as a special case of a membrane intersecting a string. However, this solution lacks the essential feature that we discussed previously, namely that there was a generalisation to a metric with an independent harmonic function associated with each of the intersecting extended objects. The construction of such a generalisation for the metric (4.2) (which itself can be diagonally reduced to a single-scalar black hole in $D = 8$) would be equivalent to constructing black holes in $D = 8$ with independent charges for two of the three field strengths F^{12}, F^{13} and F^{23}. However, no simple solution of the standard multi-scalar-type with two independent charges exists [13]. Thus the extra dimension on which the membrane lies cannot be elevated to a spatial string worldsheet dimension, and hence it seems to be unnatural to regard the metric (4.2) as that of a membrane intersecting a string. It is worth remarking that there also exists an isolated two-charge (non-supersymmetric) black hole solution in $D = 8$ where the two charges are equal, corresponding to $\Delta = 3$ [12]. The oxidation of this solution to $D = 11$ again might be viewed as a membrane intersecting a string. However, in this case the three internal coordinates z_1, z_2 and z_3 enter the metric symmetrically. Since there is no generic two-charge solution that permits an extrapolation away from this degenerate configuration, there is again no compelling reason to regard such a metric configuration as a membrane intersecting a string. Indeed, this is consistent with the fact that there is no fundamental string in $D = 11$.

An analogous analysis can be applied also to the multi-scalar black hole in $D = 5$ with three independent charges q_{12}, q_{34} and q_{56} carried by the 2-form field strengths F^{12}, F^{34} and F^{56} [12]. Oxidation of this black hole to $D = 11$ gives rise to three [16, 18] intersecting membranes. If all the charges are equal, the resulting $\Delta = 4/3$ Reissner-Nordstrøm black hole in $D = 5$ becomes a Güven solution in $D = 11$. If one of the charges is zero, it becomes two multi-center intersecting membranes with the remaining two charges uniformly distributed over a plane. When two of the charges are zero, it becomes the standard multi-center membrane with the remaining charge uniformly distributed over a 4-dimensional

hyperplane.

We may construct a further example by starting from the multi-scalar black hole in $D = 3$ [9], with four independent electric charges q_{12}, q_{34}, q_{56} and q_{78} carried by the 2-form field strengths F^{12}, F^{34}, F^{56} and F^{78}. The metric in $D = 3$ takes the form $ds^2 = -dt^2 + H_{12}H_{34}H_{56}(dy_1^2 + dy_2^2)$, where $H_{ij} = 1 + q_{ij}\log r$ and $r^2 = y_1^2 + y_2^2$. We find that dimensional oxidation of this solution to $D = 11$ gives rise to four intersecting membranes, with the metric

$$
\begin{aligned}
ds^2 \;=\; & -(H_{12}H_{34}H_{56}H_{78})^{-2/3}dt^2 + H_{12}^{-2/3}(H_{34}H_{56}H_{78})^{1/3}(dz_1^2 + dz_2^2) \\
& + H_{34}^{-2/3}(H_{12}H_{56}H_{78})^{1/3}(dz_3^2 + dz_4^2) + H_{56}^{-2/3}(H_{12}H_{34}H_{78})^{1/3}(dz_5^2 + dz_6^2) \quad (4.3)\\
& + H_{78}^{-2/3}(H_{12}H_{34}H_{56})^{1/3}(dz_7^2 + dz_8^2) + (H_{12}H_{34}H_{56}H_{78})^{1/3}(dy_1^2 + dy_2^2)\,,
\end{aligned}
$$

Thus the Güven solutions, which have $p = 2, 4, 6$ and preserve $2^{-p/2}$ of the supersymmetry, can be extended to include $p = 8$, preserving $1/16$ of the supersymmetry. This case arises when all four charges q_{12}, q_{34}, q_{56} and q_{78} are equal, corresponding to a $\Delta = 1$ black hole in $D = 3$, which again preserves $1/16$ of the supersymmetry [9].

In summary, we have seen that in $D = 11$ one can construct up to four orthogonally intersecting membranes, and that they reduce to black holes in $D = 9, 7, 5$ and 3 dimensions respectively. This is closely related to the fact that one needs 2-form field strengths with non-overlapping internal indices, for example F^{12}, F^{34}, *etc*, in order to be able to construct the necessary multi-scalar solutions with independent charges [13]. For N intersecting membranes, there are N independent charges. If n of the N charges are zero, the solution becomes $(N - n)$ multi-center intersecting membranes whose charge lies uniformly on a $2n$-dimensional hyperplane, providing the first stage of the vertical-dimensional reduction of the solution. Thus we see that some stainless p-brane solitons that are vertical-dimensional reductions of p-brane in higher dimensions can also be viewed as special cases of intersecting p-branes in the higher dimension. The criterion in general for being able to interpret a p-brane in a lower dimension as an intersection of extended objects in $D = 11$ is that there must exist corresponding multi-charge p-brane generalisations in the lower dimension. All the examples we discussed above involved elementary solutions, carrying electric charges. A completely analogous discussion applies to solitonic solutions, which carry magnetic charges.

5 Reduction trajectories linked by duality

In the vertical dimensional reduction procedure, the rank of the participating field strength remains the same for elementary solutions, whilst it is reduced by 1 in each step of the

reduction for solitonic solutions. Precisely the opposite happens under double-dimensional, *i.e.* diagonal, reduction. If one has an n-index field strength F_n in D dimensions, its Kaluza-Klein dimensional reduction gives rise to field strengths G_n and G_{n-1} in $(D-1)$ dimensions. Each of these can give rise to an elementary or a solitonic isotropic p-brane in the lower dimension. By making use of both the double dimensional and vertical dimensional reduction procedures, all four of the isotropic solutions in $(D-1)$ dimensions can be obtained from the two isotropic solutions involving F_n in D dimensions. Specifically, the elementary $(n-3)$-brane using G_{n-1} and the solitonic $(D-n-3)$-brane using G_n can be obtained by double dimensional reduction, whilst the solitonic $(D-n-2)$-brane using G_{n-1} and the elementary $(n-2)$-brane using G_n can be obtained by vertical dimensional reduction. In summary, consider a $(D-1)$-dimensional supergravity that is obtained by Kaluza-Klein dimensional reduction from a D-dimensional supergravity. *All* of the isotropic p-brane solutions in $(D-1)$ dimensions using field strengths other than the 2-form arising from the metric can be obtained either by double dimensional reduction or by vertical dimensional reduction of the isotropic solutions one dimension higher in D dimensional supergravity.

When the asymptotic values of the scalar fields are taken to vanish, the p-brane solutions with a given value of p form representations of the Weyl group [9] of the Cremmer-Julia $E_{r(+r)}$ supergravity symmetry group, $r = 11 - D$. The Weyl-group duality transformations in D dimensions rotate pure electric and pure magnetic solutions into each other, but do not rotate them into dyons (in even dimensions where dyons exist). Such transformations can now be used on the various results of double and vertical dimensional reduction when the scalar moduli take special values. For example, the elementary string and solitonic 5-brane in (D=10) are built with the same 3-form field strength. Nonetheless, there is not known to date any symmetry of $D = 10$ supergravity that directly changes the string into the 5-brane. However, upon vertical reduction of the string down to $D = 6$, one finds, for vanishing scalar moduli, a string that lies in a representation of the Weyl group of $SO(5,5)$, which is the $D = 6$ Cremmer-Julia symmetry group. This Weyl group *does* now contain a *bone fide* symmetry transformation that dualises the 3-form field strength and yields the solitonic string that descends from the $D = 10$ 5-brane by diagonal dimensional reduction. Whether this procedure of linking reduction/oxidation trajectories with dualities actually indicates the existence of a symmetry in $D = 10$ supergravity is open to debate. Dimensional reduction of supergravity solutions rely on special properties of those solutions, such as the existence of Killing vectors or obeying a zero-force condition that allows them to be stacked up. Nonetheless, the analogous states in superstring theories are widely anticipated to be

linked by non-perturbative duality symmetries, with the $D = 10$ string/5-brane relation being a familiar example.

A similar example of a duality linkage combined with the two types of dimensional reduction concerns the $D = 11$ elementary membrane and solitonic 5-brane. In $D = 11$, both of these solutions employ the same field F_4 field strength. Upon vertical reduction of the membrane to $D = 8$, one obtains a membrane that lies in the same multiplet of $S_2 \times S_3$ (the Weyl group of the $D = 8$ Cremmer-Julia group $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$) as the solitonic membrane descending from the solitonic 5-brane in $D = 11$. Here too, the transformation dualises the relevant F_4 field strength. By the same token as in the $D = 10$ case, this suggests the existence of a duality symmetry of $D = 11$ supergravity (or at least M theory) that maps the membrane and the 5-brane into each other.

Duality symmetries also exist that link p-branes using different field strengths. For example, consider a $D = 10$ elementary string and elementary membrane. In type IIA supergravity in $D = 10$, there is a single NS-NS sector F_3 field strength, which gives rise to the string, and a single R-R sector F_4 field strength, which gives rise to the membrane. Upon dimensional reduction of the $D = 10$ theory to $D = 9$, two three forms arise, one from the $D = 10$ F_3 and one from the F_4. These two $D = 9$ three-forms, (F_3^1, F_3^2) form a doublet under the $D = 9$ duality Weyl group S_2 [9]. The corresponding duality transformation maps the $D = 9$ string arising by vertical reduction from the string in $D = 10$ into the $D = 9$ string arising by diagonal reduction from the $D = 10$ membrane. If lower-dimensional symmetry indicates the existence of a higher-dimensional symmetry, this would suggest string/membrane duality in $D = 10$.

The solitonic analogue of the previous example maps the vertical reduction of the solitonic 4-brane in $D = 10$ into the diagonal reduction of the solitonic 5-brane in $D = 10$. While these two examples do not reveal electric/magnetic (or strong/weak coupling) duality (which is not relevant in $D = 9$), they reveal another characteristic of the unifying role of duality transformations, in that they link solutions using field strengths arising in the NS-NS and R-R sectors of superstring theory.

One may ask why the duality transformations in these examples are taken from the Weyl group of the Cremmer-Julia symmetry, and not from the full group. As discussed in [9], when the asymptotic values of the scalar fields vanish, the Weyl group arises as the U-duality little group of the scalar field vacuum, *i.e.* the subgroup of the full $E_r(+r)(\mathbb{Z})$ group that leaves unchanged the asymptotic values of the scalar fields. The part of $E_r(+r)$ that acts nontrivially on these moduli is akin to the set of general coordinate transforma-

tions that do not leave the asymptotic form of the metric invariant. The interpretation of such transformations may vary according to the context. In general relativity, such transformations are not customarily regarded as generating distinct new solutions, but merely are seen as giving equivalent forms of known solutions written in new coordinates. For example, flat space written in Cartesian or polar coordinates is regarded as being one and the same solution.

On the other hand, the little group of the asymptotic metric is normally treated differently. For asymptotically flat spaces, this is the Poincaré group, whose transformations are *not* treated as yielding members of an equivalence class to be identified. In fact, if one were to try to insist on a "local" interpretation for such asymptotic symmetries of the spacetime, one would have to restrict attention to the Poincaré-invariant subclass of solutions, *i.e.* those with vanishing total energy-momentum. Such a restriction is physically unwarranted; instead, one treats the Poincaré group as a rigid symmetry that *classifies* inequivalent solutions into multiplets according to their Poincaré eigenvalues.

Without taking a firm position on the interpretation of duality transformations that act nontrivially on the supergravity moduli, we would nonetheless argue that the U-duality-symmetry little group should be interpreted similarly to the asymptotic symmetry group in general relativity. The asymptotic values of the scalar fields taken together with the asymptotic values of the metric and of the antisymmetric-tensor gauge fields constitute the full set of moduli of the theory.

Classically, the manifold in which the supergravity scalar fields take their values is a coset space G/H, where G is the relevant Cremmer-Julia symmetry group $E_{r(+r)}$ and H is its linearly-realised subgroup, generally the automorphism group of the supersymmetry algebra. At this classical level, the little group of the scalar moduli is simply H. When one takes into account the quantum Dirac-Schwinger-Zwanziger charge quantisation condition, the electric and magnetic charges of p-branes are restricted to lie on a charge lattice. The subgroup of G that respects this charge quantisation is $G(\mathbb{Z})$ [10]. In the context of string theory, the U-duality symmetry $G(\mathbb{Z})$ is interpreted as a gauge symmetry, identifying vacua differing by $G(\mathbb{Z})$ transformations so that the scalar fields take their values in the manifold $G(\mathbb{Z})\backslash G/H$. In the special case of vanishing scalar moduli, the embedding of $G(\mathbb{Z})$ into G is particularly simple. Moreover, $G(\mathbb{Z})$ then has a non-trivial intersection $G(\mathbb{Z}) \cap H$ with the little group H of the scalar moduli; this intersection is in fact the Weyl group of the Cremmer-Julia symmetry G [9].

When one moves away on $G(\mathbb{Z})\backslash G/H$ from the special point with vanishing scalar

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moduli, both the embedding of $G(\mathbb{Z})$ within G and the embedding of H within G change, in both cases by conjugations involving the G transformation moving the moduli to the new point. These changes of embedding by conjugation move the two subgroups oppositely, however. Consequently, the two groups do not "track" together as one moves away from the special point on the scalar modulus space, and so $G(\mathbb{Z}) \cap H$ becomes smaller than the Weyl group of G. In fact, for generic scalar moduli on $G(\mathbb{Z}) \backslash G / H$, this intersection will be trivial, *i.e.* the identity element alone. Consequently, the Weyl group of G is the *maximal* U-duality little group, achieved at the special point of vanishing scalar moduli.

6   Modular-invariant 1-form solutions

In the above discussion, we have not advocated treating the U-duality group $G(\mathbb{Z})$ as an *active* gauge symmetry, which would require *invariance* under $G(\mathbb{Z})$. To do so would for example rule out the standard p-brane solutions, since they form non-trivial Weyl-group multiplets. Nonetheless, it is interesting to enquire whether any solutions would survive if one were to impose the restriction of full active U-duality invariance.

We shall find that one may indeed construct a special class of duality-invariant solutions, once again in the context of $(D-3)$ branes. This class will be invariant under an $SL(2, \mathbb{Z})$ duality symmetry, which in $D = 10$ type IIB supergravity is the full $G(\mathbb{Z})$ duality group. In [22], a 7-brane solution [23] of this kind was used as a background for compactification of the type IIB string to a theory in $D = 8$ that is dual to the heterotic string compactified on T^2. (The fact that this solution leads to a compactification of the transverse dimensions is again a consequence of the conical asymptotic structure of transverse spacetime for $(D-3)$ branes.) Since the 7-brane solution is invariant under the $SL(2, \mathbb{Z})$ symmetry, the transverse compactification preserves this symmetry, and this fact is crucial for establishing a duality relation with the heterotic string compactified on a torus. The 7-brane solution in $D = 10$ type IIB supergravity preserves $\frac{1}{2}$ of the supersymmetry, corresponding to the Lagrangian (1.2) with $\Delta = 4$. The form of this solution is very different from the usual $(D-3)$-brane soliton, and exploits special features of the two-dimensional transverse space, which may be viewed as a complex plane on which modular-invariant functions can be defined. As with the usual $(D-3)$-brane solution discussed above, the modular-invariant solution allows only a finite number of centers before the transverse space becomes compact. In fact for regularity of such a compactified solution, there must be exactly 24 centers [24].

In lower dimensional supergravities, where the number of 1-form field strengths pro-

liferates, we can have supersymmetric solutions involving k field strengths, giving rise to $\Delta = 4/k$. In this section, we shall show that one can in fact construct a p-brane solution that is $SL(2, \mathbb{R})$ invariant for any value of Δ. We shall then show that such a solution must have exactly 6Δ centers in order to obtain a regular compactification of the transverse space. The relevant part of the Lagrangian, describing the metric, dilaton and 1-form field strength in D dimensions is given by (1.2) with $n = 1$. The Lagrangian can be cast into the form

$$e^{-1}\mathcal{L} = R - \frac{2}{a^2}\frac{|\partial\tau|^2}{\tau_2^2} \ , \tag{6.1}$$

where $\tau = \tau_1 + i\tau_2 = \frac{a}{2}\chi + ie^{-\frac{a}{2}\phi}$, and the χ is the 0-form potential for the 1-form field strength. The dilaton prefactor for this field strength has $a^2 = \Delta$, since $\tilde{d} = 0$. Thus when $\Delta = 4$, the above Lagrangian coincides with the one discussed in [24, 23, 22]. The Lagrangian (6.1) has an $SL(2, \mathbb{R})$ symmetry

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d} \ , \tag{6.2}$$

where $ad - bc = 1$. The ansatz for the metric is given by $ds^2 = dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B}dzd\bar{z}$, where B is a function of $z = y_1 + iy_2$ and $\bar{z} = y_1 - iy_2$. The equation of motion for τ is given by

$$\partial\bar{\partial}\tau + \frac{2\partial\tau\bar{\partial}\tau}{\bar{\tau} - \tau} = 0 \ . \tag{6.3}$$

Following [24], we make the holomorphic ansatz $\bar{\partial}\tau = 0$, and hence (6.3) implies that $\partial\bar{\partial}\tau = 0$. The metric equation, after imposing this ansatz, becomes

$$\partial\bar{\partial}B = \tfrac{2}{\Delta}\partial\bar{\partial}\log\tau_2 \ , \tag{6.4}$$

The usual type of 1-form p-brane solutions discussed in the previous section correspond to taking $\tau = i\lambda\log z$ and $B = (2/\Delta)\log\tau_2 = -\phi/\sqrt{\Delta}$. Such solutions are obviously not invariant under $SL(2, \mathbb{R})$, since B is in that case a function of τ_2 only. In [24], an $SL(2, \mathbb{R})$-invariant solution was constructed for the case $\Delta = 4$. For other values of Δ, it follows from (6.3) and (6.4) that the solution for τ is unchanged, while the solution for B is simply rescaled by a factor of $4/\Delta$. Thus following [24], the asymptotic behaviour of B at large $|z|$ is given by

$$e^{2B} \sim (z\bar{z})^{-N/(3\Delta)} \ . \tag{6.5}$$

The exact invariant solution is given by

$$e^{2B} = \tau_2\eta^2\bar{\eta}^2\left|\prod_{i=1}^{N}(z - z_i)^{-1/(3\Delta)}\right|^2 \ , \tag{6.6}$$

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where $\eta = e^{i\pi\tau/12} \prod_n (1 - e^{i2n\pi\tau})$ is the Dedekind η function. It was shown in [24], for the case $\Delta = 4$, that each center contributes a deficit angle $\frac{\pi}{6}$ in the transverse space, and thus it remains non-compact if $N \leq 11$. In order to be regular and compact, one must take $N = 24$ [24]. We can therefore immediately generalise these results to the other values of Δ. In particular, there must be $N = 6\Delta$ centers in order to achieve a regular compactified transverse space.

We shall now discuss the supersymmetry of these modular-invariant solutions. First we consider the supersymmetry of the 7-brane in type IIB supergravity in $D = 10$, where (6.1) with $a^2 = 4$ is the full scalar Lagrangian of the theory. In a bosonic background where only τ and g_{MN} are excited, the supersymmetry transformation rules for the complex spin $\frac{1}{2}$ and spin $\frac{3}{2}$ fermions are [25]

$$\delta\lambda = \frac{\tau^* - i}{\tau_2(\tau + i)}\Gamma^M \partial_M \tau \epsilon^* , \qquad \delta\psi_M = \partial_M \epsilon + \tfrac{1}{4}\omega_M^{AB}\Gamma_{AB}\epsilon - \tfrac{i}{2}Q_M \epsilon , \qquad (6.7)$$

where $\epsilon = \epsilon_1 + i\epsilon_2$, $\epsilon^* = \epsilon_1 - i\epsilon_2$, and the composite $U(1)$ gauge potential is given by

$$Q_M = -\frac{1}{4\tau_2}\left\{\frac{\tau - i}{\tau^* - i}\partial_M \tau^* + \frac{\tau^* + i}{\tau + i}\partial_M \tau\right\} . \qquad (6.8)$$

Note that Q_M is invariant under the $SL(2, \mathbb{R})$ transformation (6.2), modulo a compensating local $U(1)$ gauge transformation.

The only non-vanishing spin connection, in the 7-brane metric $ds^2 = dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B}dy^m dy^m$, is given by $\omega_z^{12} = i\partial B$, $\omega_{\bar{z}}^{12} = -\bar{\partial}B$, where 1 and 2 refer to flat indices in the two-dimensional transverse space. We see that if ϵ is chosen so that $\Gamma^1\epsilon = i\Gamma^2\epsilon$, then $\Gamma^z\epsilon^* = 0$, and hence $\delta\lambda = 0$ by virtue of the holomorphicity condition $\bar{\partial}\tau = 0$. For the variation $\delta\psi_M$ to vanish also, we must have $\omega_M^{12} = Q_M$, up to a local $U(1)$ gauge transformation. Thus we must have $i\partial B = \widetilde{Q}_z \equiv Q_z + \partial f(\tau)$, where f is a holomorphic function that must be added to Q_z in order to make $\widetilde{Q}_z$ $SL(2, \mathbb{R})$ invariant. The function f has to satisfy

$$\partial f(\tau') - \partial f(\tau) = Q_z(\tau) - Q_z(\tau') = -\partial[(2i\log\left((\tau + i)(\tau + \tfrac{b+id}{a+ic})\right)] . \qquad (6.9)$$

One way to solve this is to take $f(z) = -i\log[\partial(\tau + i)^3]$.

It is easy to verify that the condition $i\partial B = \tilde{Q}_z$ that is required for supersymmetry yields (6.4) after acting on it with $\bar{\partial}$. Thus this supersymmetry condition is the $SL(2, \mathbb{R})$-invariant first integral of the equation of motion for the metric. The $SL(2, \mathbb{R})$-invariant expression (6.6) (with $\Delta = 4$) for the 7-brane is a solution of this first integral.

As we mentioned above, the 1-form field strength in $D = 10$ type IIB supergravity can also give rise to a 7-brane of the usual $SL(2, \mathbb{R})$ non-invariant kind, with $\tau = i\lambda \log z$

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and $B = \frac{1}{2}\log\tau_2$. However, this solution is not supersymmetric since, as we saw above, supersymmetry requires not only that τ be a holomorphic function but also that B be $SL(2,\mathbb{R})$ invariant (up to a $U(1)$ gauge transformation). In $D \leq 9$, on the other hand, the $SL(2,\mathbb{R})$ non-invariant $(D-3)$-brane solutions can be supersymmetric [11]. The first example is the isotropic 6-brane in $D = 9$ [11], which preserves $\frac{1}{2}$ of the supersymmetry. Since this solution is not $SL(2,\mathbb{R})$ invariant, it is not related to the above supersymmetric 7-brane in type IIB supergravity by dimensional oxidation. It was accordingly called a stainless 6-brane in [11]. On the other hand, double dimensional reduction of the 7-brane in type IIB supergravity will give rise to their own trajectory of supersymmetric $SL(2,\mathbb{R})$ invariant $(D-3)$-branes in lower dimensions. This $SL(2,\mathbb{R})$ symmetry is the full symmetry group in type IIB supergravity. In lower dimensions, the symmetry group of the supergravity theories enlarges, and so these solutions are invariant only under a subgroup of the full symmetry group. The question naturally arises whether one can construct further $(D-3)$-brane solutions that are invariant under the larger symmetries.

In lower-dimensional supergravities, the number of axions proliferates, and $(D-3)$-branes can be constructed using multiple field strengths. The relevant part of the Lagrangian can be reduced to (6.1) in a manner that is consistent with the equations of motion from the full Lagrangian [12]. In particular, we can construct the usual supersymmetric solutions with up to $k = 7$ 1-form field strengths in $D = 4$, and up to $k = 8$ in $D = 3$ [12]. The values of Δ for these supersymmetric solutions are given by $\Delta = 4/k$. We have seen in this section that such field configurations can also be used to construct $SL(2,\mathbb{R})$-invariant solutions. The supersymmetry of these solutions requires further investigation; however, we expect that they preserve the same fraction of the supersymmetry as their $SL(2,\mathbb{R})$ non-invariant counterparts.

Acknowledgements

Note added

After this paper was circulated, we learned that the solution for four intersecting membranes was also constructed in [18]. The vertical dimensional reduction of $(D-3)$-branes, and the

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relation to massive supergravities, has been studied recently in [26].

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